Lecture10: Root Locus Technique (Part1)

10.1 Background

In the previous chapters we have seen that the stability of any closed loop system depends on the locations of the roots of the characteristic equation i.e. the locations of closed loop poles. Nature of the transient response is closely related to the location of the poles in the s-plane. It is advantageous to know how the closed loop poles move in the s-plane if some parameters of the system are varied. The knowledge of such movement of the closed loop poles with small changes in the parameters of the system greatly helps in the design of any closed loop system.

Such movement of the poles can be known by the **Root Locus method**, introduced by W. R. Evans in 1948. This is a graphical method, in which movement of poles in the s-plane is sketched when a particular parameter of system is varied from zero to infinity. Note that the parameter is usually the gain but any other parameter may be varied. But for root locus method, **gain** is assumed to be a parameter which is to be varied from zero to infinity.

10.2 Basic Concept of Root Locus

In general, the characteristic equation of a closed loop system is given as,

$$1 + G(s)H(s) = 0$$

For root locus, the gain 'K' is assumed to be a variable parameter and is a part of forward path of the closed loop system. Consider the system shown in the Fig.10.1.

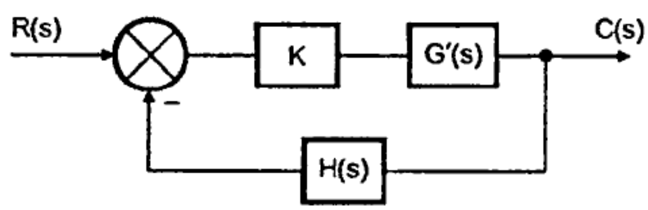


Fig.10.1

$$G(s) = KG'(s)$$

where K = Gain of the amplifier in forward path or also called **System Gain.** The characteristic equation becomes,

$$1 + G(s)H(s) = 0 \quad \text{i.e.} \quad 1 + KG'(s)H(s) = 0$$

which contains 'K' as a variable parameter.

Key Point: *The closed loop poles i.e. the roots of the above equation are now dependent on the values of 'K'.*

If now gain 'K' is varied from $-\infty$ to $+\infty$ then for each separate value of 'K' we will get separate set of locations of the roots of the characteristic equation. If all such locations are joined, the resulting locus is called **Root Locus.** So we can define root locus as, **the locus of the closed loop poles obtained when system gain 'K' is varied from $-\infty$ to $+\infty$ is called Root Locus.**

Key Point: *When `K' is varied from 0 to + ∞, the plot is called* ***Direct root locus*** *while when `K' is varied from – ∞ to 0, the plot obtained is called* ***Inverse root locus.***

But generally the term root locus is used in the sense of Direct root locus. Unless otherwise stated, the variation in gain K is assumed to be 0 to + ∞ and plot is called Root Locus.

10.3 Construction of Root Locus

To understand the rules of construction of root locus for higher order systems, let us examine some simple systems and draw some important conclusions.

Example 10.1 : *Consider unity feedback system with* $G(s) = \frac{K}{s}$. *Obtain its roots locus.*

Solution : The characteristic equation becomes,

$$1 + G(s)H(s) = 0, \quad H(s) = 1$$

$$\therefore \quad 1 + \frac{K}{s} = 0$$

$$\therefore \quad s + K = 0$$

The root of this equation is located at s = – K

Now if gain `K' is varied from 0 to + ∞, the location of this root is going to change.

The locus obtained by joining all such locations when K is varied from 0 to + ∞ is called Root Locus.

K	s = – K Root location
0	0
1	– 1
10	– 10
⋮	⋮
+ ∞	– ∞

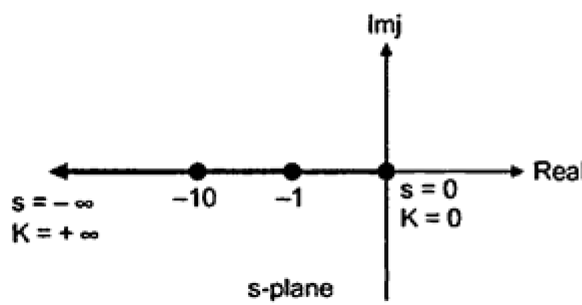


Fig.10.2

The root locus is nothing but the negative real axis, for this system.

In example 10.1, $G(s)H(s) = \frac{K}{s}$, we have seen that there is one branch of the root locus. Branch is starting at s = 0 which is open loop pole location while it is terminating at s = – ∞ .

Example10.2: *For* $G(s)H(s) = \dfrac{K}{s(s+2)}$, *obtain the nature of the root locus.*

Solution : $1 + G(s)H(s) = 1 + \dfrac{K}{s(s+2)} = 0$

i.e. $s^2 + 2s + K = 0$

Solving for its roots, roots $= \dfrac{-2 \pm \sqrt{4-4K}}{2} = -1 \pm \sqrt{1-K}$

Root 1 say $s_1 = -1 + \sqrt{1-K}$ and Root 2 say $s_2 = -1 - \sqrt{1-K}$. Let us see the locations for various values of K.

K	$s_1 = -1 + \sqrt{1-K}$	$s_2 = -1 - \sqrt{1-K}$
0	0	−2
0.2	−0.105	−1.895
0.8	−0.552	−1.448
1	−1	−1
5	−1 + j2	−1 − j2
:	:	:
∞	−1 + j∞	−1 − j∞

This root locus has two branches, one showing locus or movement of s_1 while second showing movement of s_2. Both the branches are approaching to −1 and then breaking into two, moving parallel to imaginary axis. This is shown in Fig.10.3.

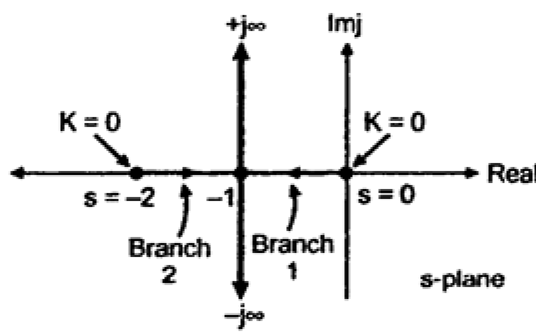


Fig.10.3

From this we can conclude that there are two branches. Branches are starting from s = 0 and s = −2 which are open loop poles of the system. Both the branches are approaching to infinity. Hence in general we can conclude that **number of branches equals number of open loop poles.** A separate branch starts from each location of open loop pole. Now to confirm whether branches always terminate at infinity or not let us see root locus of another system.

Example10.3: *Consider* $G(s)H(s) = \dfrac{K(s+1)}{s(s+5)}$. *Obtain the nature of its root locus.*

Solution : The characteristic equation is ,

$$1 + G(s)H(s) = 1 + \frac{K(s+1)}{s(s+5)} = 0$$

i.e. $s^2 + 5s + Ks + K = 0$

i.e. $s^2 + s(K+5) + K = 0$

Roots are $\dfrac{-(K+5) \pm \sqrt{(K+5)^2 - 4K}}{2} = \dfrac{-(K+5)}{2} \pm \dfrac{\sqrt{K^2 + 6K + 25}}{2}$

Effect of variation in 'K' is

K	$s_1 = \frac{-(K+5)}{2} + \frac{\sqrt{K^2+6K+25}}{2}$	$s_2 = \frac{-(K+5)}{2} - \frac{\sqrt{K^2+6K+25}}{2}$
0	0	− 5
1	− 0.1715	− 5.828
5	− 0.527	− 9.472
:	:	:
∞	− 1	− ∞

We can observe, number of branches are again two i.e. number of open loop poles. Both branches are starting from s = 0 and s = − 5 which are open loop poles. **But important observation is, one of the branches terminates at s = − 1 which is open loop zero,** while other branch is terminating at infinity.

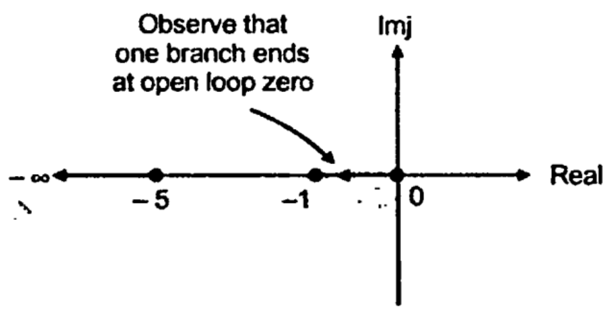


Fig. 10.4

It is very difficult to plot the root locus for higher order systems by the method of actually substituting different values of 'K' in the roots of characteristic equation as used above.

To simplify the construction of the root locus for higher order systems certain rules are specified based on the observations made earlier.

10.4 Angle and Magnitude Condition

For a general closed loop system the characteristic equation is,

$$1 + G(s)H(s) = 0$$

i.e. $$G(s)H(s) = -1$$

As s-plane is complex we can write above equation as,

$$\boxed{G(s)H(s) = -1 + j0}$$

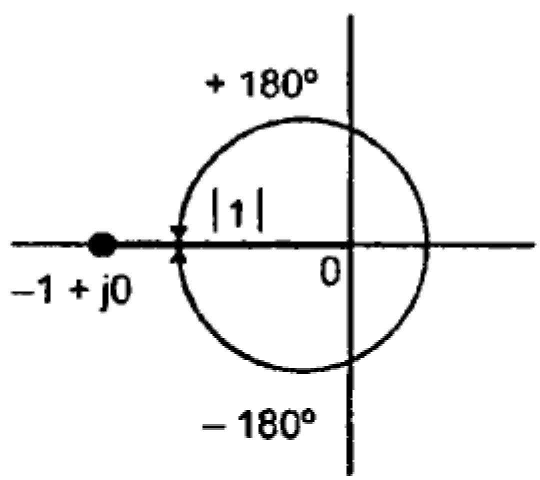
10.4.1 Angle Condition

$$G(s)H(s) = -1 + j0$$

Equating angles of both sides,

$$\angle G(s)H(s) = \pm(2q+1)\,180^\circ \qquad q = 0, 1, 2 \ldots\ldots$$

Key Point: *$-1 + j0 = 1 \angle \pm 180^\circ$ but the point $-1 + j0$ is a point on negative real axis which can be traced as magnitude 1 at an angle $\pm 180^\circ$, $\pm 540^\circ$, $\pm 900^\circ$ ..... $\pm(2q+1)\,180^\circ$.*



Magnitude = 1
Angle = $\pm 180^\circ$, $\pm 540^\circ$, - - - - - $\pm(2q+1)\,180^\circ$

Fig.10.5

Example 10.4: *Consider the system with $G(s)H(s) = \dfrac{K}{s(s+2)(s+4)}$. Find whether $s = -0.75$ is on the root locus or not using angle condition.*

Solution : Let us test whether $s = -0.75$ is located on the root locus of above system i.e. whether $s = -0.75$ is a root of the characteristic equation $1 + G(s)H(s) = 0$ or not. Use Angle condition,

$$\angle G(s)H(s)\Big|_{\text{at point } s=-0.75} = \pm(2q+1)\,180^\circ \quad q = 0, 1, 2, \ldots..$$

Substituting s = – 0.75 in all the terms of G(s)H(s),

$$\angle G(s)H(s)\Big|_{\text{at } s=-0.75} = \frac{\angle K + j0}{\angle -0.75 + j0 \cdot \angle\ 1.25 + j0 \cdot \angle\ 3.25 + j0}$$

Converting to polar form and considering angles, **(use calculator to obtain polar form from rectangular form and consider angle.)**

$$= \frac{0^\circ}{180^\circ \cdot 0^\circ \cdot 0^\circ} = -180^\circ$$

That is ∠ G(s)H(s) = – 180° at s = – 0.75 which satisfies angle condition and we can conclude that point s = – 0.75 is on the root locus of the given system.

Let us test, s = – 1 + j4 for its existence on the root locus of the same system,

$$\angle G(s)H(s)\Big|_{\text{at } s=-1+j4} = \frac{\angle K + j0}{\angle -1 + j4 \cdot \angle\ 1 + j4 \cdot \angle\ 3 + j4}$$

$$= \frac{0^\circ}{104.03^\circ \cdot 75.963^\circ \cdot 53.13^\circ}$$

$$= -233.123^\circ$$

$$\angle G(s)H(s)\Big|_{\text{at } s=-1+j4} = -233.123^\circ$$

As this is not satisfying the angle condition, the point (–1 + j4) cannot be on the root locus of the given system.

10.4.2 Magnitude Condition

If magnitudes of both sides of the equation G(s)H(s) = –1 are equated then we get a magnitude condition.

$$|G(s)H(s)| = |-1 + j0| = 1$$

Now in the function G(s)H(s), K is unknown and hence we cannot find out |G(s)H(s)| at any point in s-plane. So this condition is not suitable to check the existence of a point on the root locus. But once we know that a point in s-plane is on the root locus then it must satisfy magnitude condition also. So at that point which is known to be on the root locus by angle condition, we can find out value of K by using magnitude condition. This 'K' is value of the gain for which a known point on root locus is the root of the characteristic equation.

So magnitude condition is, $\left|G(s)H(s)\right|_{\substack{\text{at a point in s-plane} \\ \text{which is on root locus}}} = 1$

Key Point: *So magnitude condition can be used only when a point in s-plane is confirmed for its existence on the root locus by use of angle condition.*

Example 10.5: *Refer example* **10.4** *where* $G(s)H(s) = \frac{K}{s(s+2)(s+4)}$ *and* $s = -0.75$ *is confirmed to be on the root locus. Now we are interested in knowing that at what value of K,* $s = -0.75$ *is one of the roots of* $1 + G(s)H(s) = 0$*. Use the magnitude condition.*

Solution :

$$|G(s)H(s)|_{at\ s=-0.75} = 1$$

$$\frac{|K|}{|-0.75|\ |1.25|\ |3.25|} = 1$$

$$\therefore \quad K = 3.0468$$

In this case, $1 + G(s)H(s) = 0$ means $1 + \frac{K}{s(s+2)(s+4)} = 0$ i.e.

$s^3 + 6s^2 + 8s + K = 0$ is a cubic equation. But by use of angle and magnitude conditions one after the other we have decided that for $K = 3.0468$, one of the three roots is located at $s = -0.75$. The remaining two roots then can be easily obtained.

Key Point: *So root locus method also helps us to solve higher order polynomial very quickly.*